- Vibrating string:

Let us consider a flexible, homogeneous string, which undergoes relatively small transverse vibrations. We want to describe the string as follows:

$\rightarrow$ the string will be described by a function $u(x,t)$:
- the equilibrium position is •——•

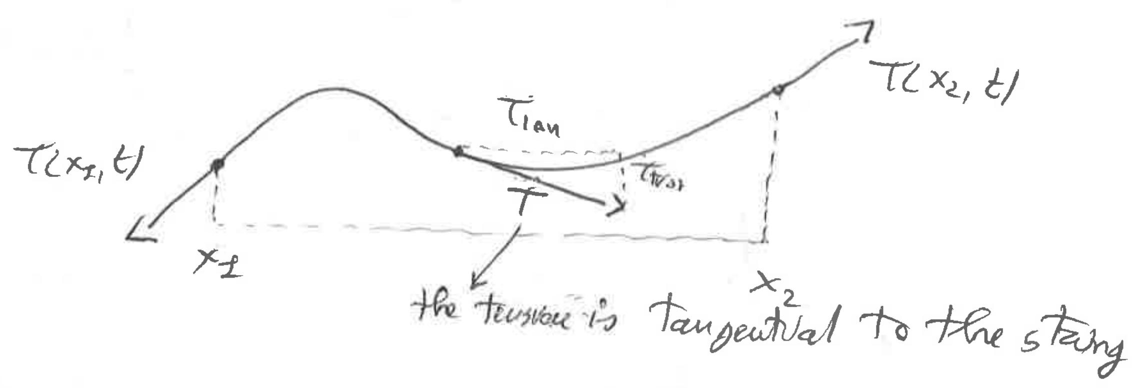
  $\Downarrow$

  $u(x,t)$ is the displacement from the equilibrium position
- it is a graph over the equilibrium position [$\Rightarrow$ no crazy horizontal movements]

Because we assumed perfect flexibility, the tension (the force) felt by the string is directly tangential along the string.

To write an equation for u, we reason as follows: we fix two arbitrary points x_1, x_2, and we write down Newton's law for that portion of the string.

Let ρ be the density of the string, and let T be the



$$F = ma \quad [\text{vectorial!}]$$

$$\Downarrow$$

in the transverse direction : (1) $T_{tra}(x_2,t) - T_{tra}(x_1,t) = \int_{x_1}^{x_2} \rho\, u_{tt}(x,t)$

in the longitudinal direction : (2) $T_{lon}(x_2,t) - T_{lon}(x_1,t) = 0$

↳ no longitudinal motion

Notice that:

(3) $T_{tra} = T \dfrac{u_x}{\sqrt{1+u_x^2}}$

(4) $T_{lon} = T \dfrac{1}{\sqrt{1+u_x^2}}$

We assume the motion is so small that $|u_x| \sim 1$. Then, by

(5) $\sqrt{1+u_x^2} \sim 1 + \frac{1}{2}|u_x|^2 \sim 1.$

So, we get:

- (2) + (4) + (5) $\Rightarrow$ $T(x_2,t) = T(x_1,t)$, and since this is valid $\forall x_1, x_2$, it means

$T(x,t) = \tilde{T}(t)$ [T is constant on the string]

We will also assume $\tilde{T}(t) \equiv c$

$$\Downarrow$$

T is constant along the string and over time

- (1) + (3) + (5) $\Rightarrow$ $$T u_x(x_2,t) - T u_x(x_1,t) = \int_{x_1}^{x_2} \rho\, u_{tt}(x,t)\, dx$$

Now, keep x_1 fixed and derive the above equation w.r.t. x_2 [that we will now call x]

$$(T u_x(x,t))_x = \rho u_{tt}(x,t)$$

T is constant $\longleftarrow$ $\parallel$

$$T u_{xx}(x,t)$$

By setting, $c := \sqrt{\frac{T}{\rho}}$, we get the equation

$$\boxed{u_{tt} - c^2 u_{xx} = 0}$$ the wave equation

- Some variants:

i) when an air resistance r is present:

$$u_{tt} - c^2 u_{xx} + r u_t = 0$$

ii) if there is a transverse elastic force:

$$u_{tt} - c^2 u_{xx} + K u = 0$$

iii) if an external force is applied:

$$u_{tt} - c^2 u_{xx} = f(x,t).$$